

# First-order logic

## CHAPTER 7

# Outline

- ◇ Syntax and semantics of FOL
- ◇ Fun with sentences
- ◇ Wumpus world in FOL

## Syntax of FOL: Basic elements

Constants *KingJohn, 2, UCB, ...*

Predicates *Brother, >, ...*

Functions *Sqrt, LeftLegOf, ...*

Variables *x, y, a, b, ...*

Connectives  $\wedge \vee \neg \Rightarrow \Leftrightarrow$

Equality  $=$

Quantifiers  $\forall \exists$

## Atomic sentences

Atomic sentence =  $\textit{predicate}(\textit{term}_1, \dots, \textit{term}_n)$   
or  $\textit{term}_1 = \textit{term}_2$

Term =  $\textit{function}(\textit{term}_1, \dots, \textit{term}_n)$   
or *constant* or *variable*

E.g.,  $\textit{Brother}(\textit{KingJohn}, \textit{RichardTheLionheart})$   
>  $(\textit{Length}(\textit{LeftLegOf}(\textit{Richard})), \textit{Length}(\textit{LeftLegOf}(\textit{KingJohn})))$

## Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g.  $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$$>(1, 2) \vee \leq(1, 2)$$

$$>(1, 2) \wedge \neg >(1, 2)$$

# Truth in first-order logic

Sentences are true with respect to a model and an interpretation

Model contains objects and relations among them

Interpretation specifies referents for

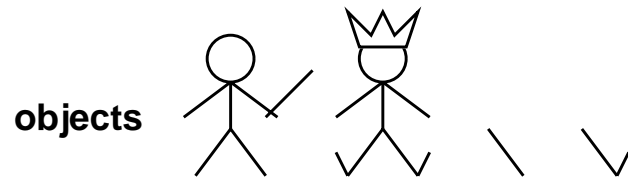
*constant symbols* → objects

*predicate symbols* → relations

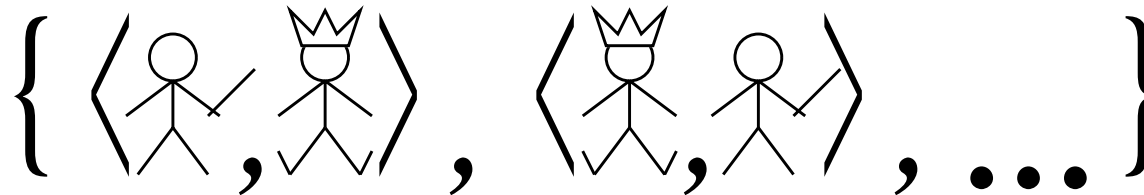
*function symbols* → functional relations

An atomic sentence  $predicate(term_1, \dots, term_n)$  is true  
iff the objects referred to by  $term_1, \dots, term_n$   
are in the relation referred to by  $predicate$

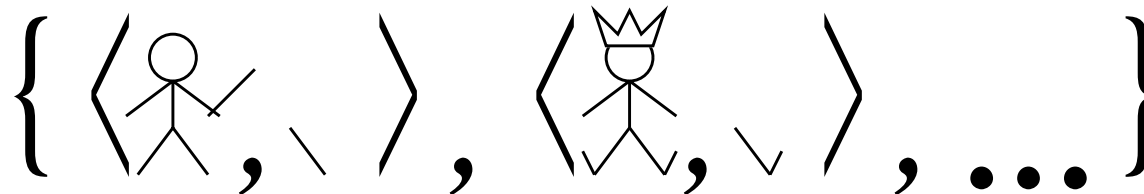
# Models for FOL: Example



**relations: sets of tuples of objects**



**functional relations: all tuples of objects + "value" object**



# Universal quantification

$\forall \langle \text{variables} \rangle \langle \text{sentence} \rangle$

Everyone at Berkeley is smart:

$\forall x \text{ At}(x, \text{Berkeley}) \Rightarrow \text{Smart}(x)$

$\forall x P$  is equivalent to the conjunction of instantiations of  $P$

$\text{At}(\text{KingJohn}, \text{Berkeley}) \Rightarrow \text{Smart}(\text{KingJohn})$   
 $\wedge \text{At}(\text{Richard}, \text{Berkeley}) \Rightarrow \text{Smart}(\text{Richard})$   
 $\wedge \text{At}(\text{Berkeley}, \text{Berkeley}) \Rightarrow \text{Smart}(\text{Berkeley})$   
 $\wedge \dots$

Typically,  $\Rightarrow$  is the main connective with  $\forall$ .

Common mistake: using  $\wedge$  as the main connective with  $\forall$ :

$\forall x \text{ At}(x, \text{Berkeley}) \wedge \text{Smart}(x)$

means “Everyone is at Berkeley and everyone is smart”

# Existential quantification

$\exists \langle \text{variables} \rangle \langle \text{sentence} \rangle$

Someone at Stanford is smart:

$\exists x \text{ At}(x, \text{Stanford}) \wedge \text{Smart}(x)$

$\exists x P$  is equivalent to the disjunction of instantiations of  $P$

$\text{At}(\text{KingJohn}, \text{Stanford}) \wedge \text{Smart}(\text{KingJohn})$   
 $\vee \text{At}(\text{Richard}, \text{Stanford}) \wedge \text{Smart}(\text{Richard})$   
 $\vee \text{At}(\text{Stanford}, \text{Stanford}) \wedge \text{Smart}(\text{Stanford})$   
 $\vee \dots$

Typically,  $\wedge$  is the main connective with  $\exists$ .

Common mistake: using  $\Rightarrow$  as the main connective with  $\exists$ :

$\exists x \text{ At}(x, \text{Stanford}) \Rightarrow \text{Smart}(x)$

is true if there is anyone who is not at Stanford!

# Properties of quantifiers

$\forall x \forall y$  is the same as  $\forall y \forall x$  (why??)

$\exists x \exists y$  is the same as  $\exists y \exists x$  (why??)

$\exists x \forall y$  is not the same as  $\forall y \exists x$

$\exists x \forall y \text{ Loves}(x, y)$

“There is a person who loves everyone in the world”

$\forall y \exists x \text{ Loves}(x, y)$

“Everyone in the world is loved by at least one person”

Quantifier duality: each can be expressed using the other

$\forall x \text{ Likes}(x, \text{IceCream}) \quad \neg \exists x \neg \text{Likes}(x, \text{IceCream})$

$\exists x \text{ Likes}(x, \text{Broccoli}) \quad \neg \forall x \neg \text{Likes}(x, \text{Broccoli})$

## Fun with sentences

Brothers are siblings

.

“Sibling” is reflexive

.

One’s mother is one’s female parent

.

A first cousin is a child of a parent’s sibling

.

.

$$\forall x, y \text{ Brother}(x, y) \Leftrightarrow \text{Sibling}(x, y).$$

.

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x)$$

.

$$\forall x, y \text{ Mother}(x, y) \Leftrightarrow (\text{Female}(x) \text{ and } \text{Parent}(x, y))$$

.

$$\forall x, y \text{ FirstCousin}(x, y) \Leftrightarrow \exists p, ps \text{ Parent}(p, x) \wedge \text{Sibling}(ps, p) \wedge \text{Parent}(ps, y)$$

# Equality

$term_1 = term_2$  is true under a given interpretation  
if and only if  $term_1$  and  $term_2$  refer to the same object

E.g.,  $1 = 2$  and  $\forall x \times(Sqrt(x), Sqrt(x)) = x$  are satisfiable  
 $2 = 2$  is valid

E.g., definition of (full) *Sibling* in terms of *Parent*:

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists m, f \neg(m = f) \wedge \\ \text{Parent}(m, x) \wedge \text{Parent}(f, x) \wedge \text{Parent}(m, y) \wedge \text{Parent}(f, y)]$$

## Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at  $t = 5$ :

$\text{TELL}(KB, \text{Percept}([\text{Smell}, \text{Breeze}, \text{None}], 5))$

$\text{ASK}(KB, \exists a \text{ Action}(a, 5))$

I.e., does the KB entail any particular actions at  $t = 5$ ?

Answer: *Yes*,  $\{a/\text{Shoot}\}$  ← substitution (binding list)

Given a sentence  $S$  and a substitution  $\sigma$ ,

$S\sigma$  denotes the result of plugging  $\sigma$  into  $S$ ; e.g.,

$S = \text{Smarter}(x, y)$

$\sigma = \{x/\text{Hillary}, y/\text{Bill}\}$

$S\sigma = \text{Smarter}(\text{Hillary}, \text{Bill})$

$\text{ASK}(KB, S)$  returns some/all  $\sigma$  such that  $KB \models S\sigma$

## Knowledge base for the wumpus world

“Perception”

$\forall b, g, t \text{ Percept}([Smell, b, g], t) \Rightarrow Smelt(t)$

$\forall s, b, t \text{ Percept}([s, b, Glitter], t) \Rightarrow AtGold(t)$

Reflex:  $\forall t \text{ AtGold}(t) \Rightarrow \text{Action}(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall t \text{ AtGold}(t) \wedge \neg \text{Holding}(Gold, t) \Rightarrow \text{Action}(Grab, t)$

$\text{Holding}(Gold, t)$  cannot be observed

$\Rightarrow$  keeping track of change is essential

## Deducing hidden properties

Properties of locations:

$$\forall l, t \text{ At}(\text{Agent}, l, t) \wedge \text{Smelt}(t) \Rightarrow \text{Smelly}(l)$$

$$\forall l, t \text{ At}(\text{Agent}, l, t) \wedge \text{Breeze}(t) \Rightarrow \text{Breezy}(l)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall y \text{ Breezy}(y) \Rightarrow \exists x \text{ Pit}(x) \wedge \text{Adjacent}(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y \text{ Pit}(x) \wedge \text{Adjacent}(x, y) \Rightarrow \text{Breezy}(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the *Breezy* predicate:

$$\forall y \text{ Breezy}(y) \Leftrightarrow [\exists x \text{ Pit}(x) \wedge \text{Adjacent}(x, y)]$$

# Keeping track of change

Facts hold in situations, rather than eternally

E.g.,  $Holding(Gold, Now)$  rather than just  $Holding(Gold)$

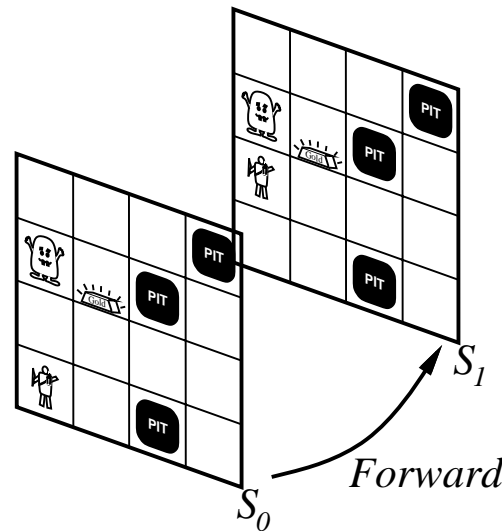
Situation calculus is one way to represent change in FOL:

Adds a situation argument to each non-eternal predicate

E.g.,  $Now$  in  $Holding(Gold, Now)$  denotes a situation

Situations are connected by the *Result* function

$Result(a, s)$  is the situation that results from doing  $a$  in  $s$



## Describing actions I

“Effect” axiom—describe changes due to action

$$\forall s \text{ AtGold}(s) \Rightarrow \text{Holding}(\text{Gold}, \text{Result}(\text{Grab}, s))$$

“Frame” axiom—describe non-changes due to action

$$\forall s \text{ HaveArrow}(s) \Rightarrow \text{HaveArrow}(\text{Result}(\text{Grab}, s))$$

Frame problem: find an elegant way to handle non-change

(a) representation—avoid frame axioms

(b) inference—avoid repeated “copy-overs” to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or ...

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, ...

## Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is “about” a predicate (not an action per se):

$$\begin{aligned} P \text{ true afterwards} &\Leftrightarrow [\text{an action made } P \text{ true} \\ &\vee P \text{ true already and no action made } P \text{ false}] \end{aligned}$$

For holding the gold:

$$\begin{aligned} \forall a, s \text{ } Holding(Gold, Result(a, s)) &\Leftrightarrow \\ &[(a = Grab \wedge AtGold(s)) \\ &\vee (Holding(Gold, s) \wedge a \neq Release)] \end{aligned}$$

## Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$

$At(Gold, [1, 2], S_0)$

Query:  $ASK(KB, \exists s \text{ Holding}(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer:  $\{s / Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at  $S_0$  and that  $S_0$  is the only situation described in the KB

## Making plans: A better way

Represent plans as action sequences  $[a_1, a_2, \dots, a_n]$

$PlanResult(p, s)$  is the result of executing  $p$  in  $s$

Then the query  $ASK(KB, \exists p \text{ Holding}(Gold, PlanResult(p, S_0)))$   
has the solution  $\{p/[Forward, Grab]\}$

Definition of  $PlanResult$  in terms of  $Result$ :

$$\forall s \text{ } PlanResult([], s) = s$$

$$\forall a, p, s \text{ } PlanResult([a|p], s) = PlanResult(p, Result(a, s))$$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

# Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB